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Macroscopic strings as heavy quarks: Large-N gauge theory and anti-de Sitter supergravity*

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Abstract. We study some aspects of Maldacena's large-N correspondence between $\mathcal{N} = 4$ superconformal gauge theory on the D3-brane and maximal supergravity on $\mathrm{AdS}_5 \times S_5$ by introducing macroscopic strings as heavy (anti-) quark probes. The macroscopic strings are semi-infinite Type IIB strings ending on a D3-brane world-volume. We first study deformation and fluctuation of D3-branes when a macroscopic BPS string is attached. We find that both dynamics and boundary conditions agree with those for the macroscopic string in anti-de Sitter supergravity. As a by-product we clarify how Polchinski's Dirichlet and Neumann open string boundary conditions arise dynamically. We then study the non-BPS macroscopic string–anti-string pair configuration as a physical realization of a heavy quark Wilson loop. We obtain the $Q\bar{Q}$ static potential from the supergravity side and find that the potential exhibits non-analyticity of the square-root branch cut in the 't Hooft coupling parameter. We put forward non-analyticity as a prediction for large-N gauge theory in the strong 't Hooft coupling limit. By turning on the Ramond–Ramond zero-form potential, we also study the θ vacuum angle dependence of the static potential. We finally discuss the possible dynamical realization of the heavy N-prong string junction and of the large-N loop equation via a local electric field and string recoil thereof. Throughout comparisons of the AdS–CFT correspondence, we find that a crucial role is played by "geometric duality" between the UV and IR scales in directions perpendicular to the D3-brane and parallel ones, explaining how the AdS_5 spacetime geometry emerges out of four-dimensional gauge theory at strong coupling.

1 Introduction

With better understanding of D-brane dynamics, new approaches to outstanding problems in gauge theory have become available. One of these problems concerns regarding the behavior of $SU(N)$ gauge theory in the large-N limit [1]: $N \to \infty$ with the 't Hooft coupling $g^2_{\mathrm{eff}} = g^2_{\mathrm{YM}} N$ fixed. Planar diagram dominance as shown first by 't Hooft has been regarded as indicative of a certain connection to string theory, but it has never become clear how and to what extent the string is related to the fundamental string. Recently, built on an earlier study of the near-horizon geometry of D- and M-branes [2] and their absorption and Hawking emission processes [3], Maldacena has put forward a remarkable proposal for the large-N behavior [4]. According to his proposal, the large-N limit of d-dimensional conformal field theories with sixteen supercharges is governed in a dual description by maximal supergravity theories (chiral or non-chiral depending on d) with thirty-two supercharges that are compactified on AdS_{d+1} times the internal round sphere. Extensions to non-conformally invariant field theories [5] and new results [6–9] extending Maldacena's proposal have been reported.

The most tractable example of Maldacena's proposal is four-dimensional $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group $SU(N)$. This theory is superconformally invariant with vanishing beta function and is realized as the world-volume theory of N coincident D3-branes of Type IIB string theory. The latter produces the near-horizon geometry of $\mathrm{AdS}_5 \times S_5$, where $\lambda_{\mathrm{IIB}} = g^2_{\mathrm{YM}}$, we have the radius of curvature $g^{1/2}_{\mathrm{eff}} \ell_{\mathrm{s}}$ and a self-dual flux $Q_5 = (1/2\pi) \int_{S_5} H_5 = N$ units. By taking $\lambda_{\mathrm{IIB}} \to 0$ while keeping g_{eff} large in the large-N limit, the classical Type IIB string theory is approximated by compactified supergravity.

In this paper, we study some aspects of the large-N behavior of superconformal $d = 4$, $\mathcal{N} = 4$ Yang–Mills theory with gauge group $SU(N)$ from the perspectives of Maldacena's proposal. In particular, we pay attention to charged particles in the theory. It is well known that conformal invariance imposes a vanishing electric current as an operator equation, leading only to a trivial theory. It has been argued that [10], to obtain a non-trivial conformally invariant fixed point, there must be non-vanishing electric and magnetic states in the spectrum. Then it would be most desirable to investigate the theory with charged

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particles in detail. Massless charged particles, even though being of our ultimate interest, would be rather delicate because their long-range fields are exponentially suppressed due to conformal invariance. Thus, in this paper, we would like to concentrate exclusively on heavy electric and magnetic particles.

The idea is very simple. The spectrum of $d = 4, \mathcal{N} = 4$ super-Yang–Mills theory contains BPS spectra carrying electric and magnetic charges (p, q). Extending Maldacena's conjecture, one expects that the correspondence between gauge theory and supergravity continues to hold even when heavy charged particles are present. In particular, the dynamics of BPS particles should match gauge theory and supergravity descriptions. On the supergravity side, charged particle may be described by a macroscopic Type IIB (p, q) string that ends on the D3-branes. For example, ending on a D3-brane, a macroscopic fundamental $(1, 0)$ string represents a static, spinless quark transforming in the defining representation of the $SU(N)$ gauge group. On the gauge theory side, one can also describe the BPS charged particles as world-volume solitons on the D3-brane. Using the Born–Infeld world-volume action, Callan and Maldacena [11] have shown that the world-volume BPS solitons are identical to the Type IIB (p, q) string ending on the D3-branes. Thus, equipped with both supergravity and world-volume descriptions, one would be able to test Maldacena's conjecture explicitly even when the conjecture is extended to include heavy charged states.

Using the aforementioned correspondence between heavy charged states and macroscopic strings, we will prove that the static quark–anti-quark potential comes out of the regularized energy of a static configuration of open Type IIB string in an anti-de Sitter supergravity background. We will find that the static potential is of Coulomb type, the unique functional form being consistent with the underlying conformal invariance [21], and, quite surprisingly, it is proportional to the *square-root* of the 't Hooft coupling parameter. We interpret the non-analyticity as an important prediction of Maldacena's conjecture on super-Yang–Mills theory in the large-N, strong 't Hooft coupling limit.

In due course of this study, we will elaborate more on boundary conditions that the world-volume BPS soliton satisfies at the throat. According to Polchinski's prescription, open string coordinates in perpendicular and parallel directions to the D-brane should satisfy Dirichlet and Neumann boundary conditions, respectively. For the world-volume BPS soliton, we will show that these boundary conditions arise quite naturally as a consequence of a self-adjoint extension [14,15] of small fluctuation operators along the elongated D3-brane world-volume of BPS soliton.

This paper is organized as follows. In Sect. 2, we study the dynamics of a macroscopic Type IIB string, using the Nambu–Goto formulation, in the background of multiple D3-branes. In Sect. 3, the result of Sect. 2 is compared with the dynamics of the Type IIB string realized as a world-volume BPS soliton on the D3-brane. We find that the two descriptions are in perfect agreement. As a bonus, we will be able to provide a dynamical account of Polchinski's D-brane boundary conditions out of the self-adjointness of the low-energy string dynamics. In Sect. 4, we also study large-N resummed Born–Infeld theory and find the D3-brane world-volume soliton that corresponds to a semi-infinite string and to massive charged particle on the D3-brane. In Sect. 5, we consider a heavy quark and anti-quark pair configuration, again, from both the large-N resummed Born–Infeld and the supergravity sides. As a prototype non-perturbative quantity, we derive the static inter-quark potential. Results from both sides are qualitatively in good agreement and, most significantly, display a surprising non-analytic behavior with respect to the 't Hooft coupling. We also point out that the static inter-quark potential suggests a dual relation between the ultraviolet (infrared) limit of the supergravity side and the infrared (ultraviolet) limit of the gauge theory side, which we refer to as UV–IR geometry duality. In Sect. 6, we speculate on the possible relevance of conformal invariance to the large-N Wilson loop equation and the realization of exotic hadron states in the large-N gauge theory via N-pronged string networks on the supergravity side.

2 String on D3-brane: Supergravity description

Consider N coincident planar D3-branes (thus carrying a total Ramond–Ramond charge $N \equiv \oint_{S_5} H_5 = \oint_{S_5} H_5^*$), all located at $\boldsymbol{x}_\perp = 0$. The supergravity background of the D3-branes is given by

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{D3}} &= G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= \frac{1}{\sqrt{G}}\left(-\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}_\parallel^2\right) + \sqrt{G}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right), \end{aligned} \tag{1}$$

where

$$G(r) = 1 + g_{\mathrm{eff}}^2\left(\frac{\sqrt{\alpha}'}{r}\right)^4. \tag{2}$$

In the strong coupling regime, $g_{\mathrm{eff}} \to \infty$, the geometry described by the near-horizon region is given by the anti-de Sitter spacetime AdS_5 times S_5. For extremal D3-branes, the dilaton field is constant everywhere. This being so, up to the string coupling factors the supergravity background (1) coincides with the string sigma-model background.

We would like to study the dynamics of a Type IIB fundamental test string that ends on the D3-branes[1]. Let us denote the string coordinates $X^\mu(\sigma, \tau)$, where σ, τ parametrize the string worldsheet. The low-energy dynamics of the test string may be described in terms of the Nambu–Goto action, whose Lagrangian is given by

$$L_{\mathrm{NG}} = T_{(n,0)}\int \mathrm{d}\sigma\sqrt{-\det h_{ab}} + L_{\mathrm{boundary}}, \tag{3}$$

[1] By $SL(2, \mathbf{Z})$ invariance of Type IIB string theory, it is straightforward to extend the results to the situation where the test string is a dyonic (p, q) string [11]

where $T_{(n,0)} = n/2\pi\alpha'$ denotes the string tension (n being the string multiplicity, which equals the electric charge on the D3-brane world-volume), L_{boundary} signifies the appropriate open string boundary condition at the location of the D3-brane, which we will discuss in more detail later, and h_{ab} is the induced metric of the worldsheet:

$$h_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu. \tag{4}$$

For the background metric $G_{\mu\nu}$, our eventual interest is the case $g_{\text{eff}} \to \infty$, so that the anti-de Sitter spacetime is zoomed in. In our analysis, however, we will retain the asymptotic flat region. Quite amusingly, from such an analysis one will be able to extend Polchinski's description of the boundary conditions for an open string ending on the D-brane in the $g_{\text{st}} = 0$ limit, where an exact conformal field theory description is valid, to an interacting string ($g_{\text{st}} \neq 0$) regime.

To find the relevant string configuration, we take $X^0 = t = \tau$ and decompose nine spatial coordinates of the string into

$$\boldsymbol{X} = \boldsymbol{X}_\parallel + \boldsymbol{X}_\perp. \tag{5}$$

Here, $\boldsymbol{X}_\parallel, \boldsymbol{X}_\perp$ represent test string coordinates longitudinal and transverse to the D3-brane. The transverse coordinates $\boldsymbol{X}_\perp$ may be decomposed further into a radial coordinate $\alpha' U$ and angular ones $\boldsymbol{\Omega}_5$. In the background metric (1), a straightforward calculation yields ($\dot{} \equiv \partial_t, {}' \equiv \partial_\sigma$)

$$\begin{aligned} h_{00} &= \sqrt{G}\dot{\boldsymbol{X}}_\perp^2 - \frac{1}{\sqrt{G}}\left(1 - \dot{\boldsymbol{X}}_\parallel^2\right), \\ h_{11} &= \sqrt{G}\left({\boldsymbol{X}'_\perp}^2\right) + \frac{1}{\sqrt{G}}{\boldsymbol{X}'_\parallel}^2, \\ h_{01} &= \frac{1}{\sqrt{G}}\dot{\boldsymbol{X}}_\parallel \cdot \boldsymbol{X}'_\parallel + \sqrt{G}\dot{\boldsymbol{X}}_\perp \cdot \boldsymbol{X}'_\perp, \end{aligned} \tag{6}$$

where $G = G(|\boldsymbol{X}_\perp|)$. From this, for a static configuration is derived the Nambu–Goto Lagrangian:

$$L_{\text{NG}} \to \int \mathrm{d}\sigma \sqrt{{\boldsymbol{X}'_\perp}^2 + \frac{1}{G}{\boldsymbol{X}'_\parallel}^2}. \tag{7}$$

From the equations of motion

$$\begin{aligned} &\left(\frac{\boldsymbol{X}'_\perp}{\sqrt{{\boldsymbol{X}'_\perp}^2 + \frac{1}{G}{\boldsymbol{X}'_\parallel}^2}}\right)' = {\boldsymbol{X}'_\parallel}^2(\nabla_{\boldsymbol{x}_\perp} G^{-1}), \\ &\left(\frac{\frac{1}{G}\boldsymbol{X}'_\parallel}{\sqrt{{\boldsymbol{X}'_\perp}^2 + \frac{1}{G}{\boldsymbol{X}'_\parallel}^2}}\right)' = 0, \end{aligned} \tag{8}$$

it is easy to see that the solution relevant to our situation is when $\boldsymbol{X}'_\parallel = 0$ (a class of solutions with $\boldsymbol{X}'_\parallel \neq 0$ corresponds to a string bent along the D3-brane, which will be treated in some detail in Sect. 4). Solving the equation for $\boldsymbol{X}_\perp$, one finds $\sigma = \alpha' U$ and $\boldsymbol{\Omega}_5$ constant. This yields precisely the static gauge configuration

$$X^0 = t = \tau, \quad \alpha' U = r. \tag{9}$$

2.1 Weak coupling limit

Consider the low-energy dynamics of the macroscopic test string in the weak coupling regime, $\lambda_{\text{IIB}} \to 0$. In this regime, the radial function part in (2) can be treated perturbatively. Expanding the Nambu–Goto Lagrangian around the static gauge configuration, (9), one derives the low-energy effective Lagrangian up to quartic order:

$$\begin{aligned} L_{\text{NG}} = \frac{T_{(n,0)}}{2}\int_0^\infty \mathrm{d}r \Bigg[\left(\dot{\boldsymbol{X}}_\parallel^2 - \frac{1}{G}{\boldsymbol{X}'_\parallel}^2\right) \\ + \left(G\dot{\boldsymbol{X}}_\perp^2 - {\boldsymbol{X}'_\perp}^2\right) + \left(\dot{\boldsymbol{X}}_\parallel \cdot \boldsymbol{X}'_\perp - \dot{\boldsymbol{X}}_\perp \cdot \boldsymbol{X}'_\parallel\right)^2\Bigg]. \end{aligned} \tag{10}$$

At the boundary $r = 0$, where the test string ends on the D3-brane, a suitable boundary condition has to be supplemented. The boundary condition should reflect the fact that the string is attached to the D3-brane dynamically and render the fluctuation wave operator self-adjoint.

Let us introduce a tortoise worldsheet coordinate σ:

$$\frac{\mathrm{d}r}{\mathrm{d}\sigma} = \frac{1}{\sqrt{G}} \equiv \cos\theta(r) \quad (-\infty < \sigma < +\infty), \tag{11}$$

in terms of which the spacetime metric (1) becomes conformally flat:

$$\mathrm{d}s^2_{\text{D3}} = \frac{1}{\sqrt{G}}\left(-\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}_\parallel^2 + \mathrm{d}\sigma^2\right) + \sqrt{G}r^2\mathrm{d}\Omega_5^2. \tag{12}$$

The quadratic part of the low-energy effective Lagrangian is

$$\begin{aligned} L_{\text{NG}} = \frac{T_{(n,0)}}{2}\int_{-\infty}^{+\infty} \mathrm{d}\sigma \Bigg[\frac{1}{\sqrt{G}}\left((\partial_t \boldsymbol{X}_\parallel)^2 - (\partial_\sigma \boldsymbol{X}_\parallel)^2\right) \\ + \sqrt{G}\left((\partial_t \boldsymbol{X}_\perp)^2 - (\partial_\sigma \boldsymbol{X}_\perp)^2\right)\Bigg], \end{aligned} \tag{13}$$

which reflects explicitly the conformally flat background (12). The Lagrangian clearly displays the fact that both parallel and transverse fluctuations propagate at the speed of light, despite the fact that both mass density and tension of the string are varying spatially.

Note that, in the tortoise coordinate of (11) and (12), $\sigma \to -\infty$ corresponds to the near D3-brane $r \to 0$, while $\sigma \to +\infty$ is the asymptotic spatial infinity $r \to \infty$. In the limit $g_{\text{eff}} \to \infty$, the boundary of anti-de Sitter spacetime is at $\sigma = 0$. Therefore, to specify dynamics of the open test string, appropriate self-adjoint boundary conditions have to be supplemented at $\sigma = -\infty$ and at $\sigma = 0$ if the anti-de Sitter spacetime is zoomed in. To analyze the boundary conditions, we now examine scattering of low-energy excitations off the D3-brane.

For a monochromatic transverse fluctuation $\boldsymbol{X}_\perp(\sigma, t) = \boldsymbol{X}_\perp(\sigma)\mathrm{e}^{-\mathrm{i}\omega t}$, the unitary transformation $\boldsymbol{X}_\perp(\sigma) \to G^{-1/4}\boldsymbol{Y}_\perp(\sigma)$ combined with a change of variables $\sigma \to \sigma/\omega, r \to r/\omega, g_{\text{eff}} \to g_{\text{eff}}/\omega$, where $\epsilon \equiv g_{\text{eff}}^{1/2}\omega$, yields the fluctuation equation into a one-dimensional Schrödinger equation form:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + V_\perp(\sigma)\right]\boldsymbol{Y}_\perp(\sigma) = +1 \cdot \boldsymbol{Y}_\perp(\sigma), \tag{14}$$

where the analog potential $V(\sigma)$ is given by

$$V_\perp(\sigma) = -\frac{1}{16}G^{-3}\left[5(\partial_r G)^2 - 4G(\partial_r^2 G)\right] = \frac{5\epsilon^{-2}}{(r^2/\epsilon^2+\epsilon^2/r^2)^3}. \qquad (15)$$

For low-energy scattering, $\epsilon \to 0$, the potential may be approximated by a δ function[2]. We now elaborate the justification of their approximation. This analog potential has a maximum at $r=\epsilon$. In terms of the σ coordinates, this is again at $\sigma \approx \mathcal{O}(\epsilon)$. We thus find that the one-dimensional Schrödinger equation has a delta function-like potential. For low-energy scattering, the delta function gives rise to the Dirichlet boundary condition. An interesting situation is when $g_{\rm eff} \to 0$. The distance between $r=0$ and $r=\epsilon$ becomes zero. Therefore, the low-energy scattering may be described by a self-adjoint extension of the free Laplacian operator at $r=0$.

Similarly, for a monochromatic parallel fluctuation $\boldsymbol{X}_\parallel(t,\sigma) = \boldsymbol{X}_\parallel(\sigma)\mathrm{e}^{-\mathrm{i}\omega t}$, the unitary transformation $\boldsymbol{X}_\parallel = G^{1/4}\boldsymbol{Y}_\parallel$ combined with the same change of variables yields

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + V_\parallel(\sigma)\right]\boldsymbol{Y}_\parallel(\sigma) = +1\cdot\boldsymbol{Y}_\parallel(\sigma), \qquad (16)$$

where

$$V_\parallel(\sigma) = \frac{1}{16}G^{-3}[7(\partial_r G)^2 - 4G(\partial_r^2 G)] = -\frac{(5r^2/\epsilon^2 - 2\epsilon^2/r^2)}{(r^2/\epsilon^2+\epsilon^2/r^2)^3}. \qquad (17)$$

By a similar reasoning as the transverse fluctuation case, for low-energy scattering $\epsilon \to 0$, it is straightforward to convince oneself that the analog potential approches $\delta'(\sigma-\epsilon)$, the derivative of the delta function potential. It is well known that the δ' potential yields the Neumann boundary condition [14,15]. An interesting point is that the scattering center is *not* at the brane location $r=0$ as would naively be thought from conformal field theory reasoning, but a distance $\mathcal{O}(\epsilon)$ away.

We have thus discovered that the Polchinski's conformal field theoretic description for boundary conditions of an open string ending on D-branes follows quite naturally from dynamical considerations of a string fluctuation in the low-energy, weak 't Hooft coupling, $g_{\rm eff} \to 0$, limit.

2.2 Strong coupling limit

Let us now consider the low-energy dynamics of the test string in the strong coupling regime, $g_{\rm eff} \to \infty$. Suppose N coincident D3-branes are located at $|\boldsymbol{x}_\perp| \equiv \ell_s^2 U = 0$ and, in this background, a probe D3-brane of charge k ($k \ll N$) is located at $\boldsymbol{x}_\perp = \boldsymbol{x}_0$. We will be considering a macroscopic fundamental Type IIB string attached to the probe D3-brane, but in the simplifying limit the probe D3-brane approaches the N coincident D3-branes. In this case, $\boldsymbol{x}_0 \to 0$, and the function $G(r)$ in (2) is reduced to

$$G = 1 + g_{\rm eff}^2\left[\left(\frac{\sqrt{\alpha'}}{r}\right)^4 + \frac{k}{N}\left(\frac{\sqrt{\alpha'}}{|\boldsymbol{x}_\perp - \boldsymbol{x}_0|}\right)^4\right] \to \frac{\widetilde{g_{\rm eff}}^2}{\alpha'}\frac{1}{U^4}, \quad \text{where} \quad \widetilde{g_{\rm eff}}^2 = \left(1+\frac{k}{N}\right)g_{\rm eff}^2. \qquad (18)$$

The resulting near-horizon geometry is nothing but $\mathrm{AdS}_5 \times S^5$ modulo rescaling of the radius of curvature. Then, the low-energy effective Lagrangian (10) becomes

$$L = \frac{T_{(n,0)}}{2}\int \mathrm{d}U\left[U^2\left(\frac{\widetilde{g_{\rm eff}}^2}{U^4}(\partial_t\boldsymbol{\Omega})^2 - (\partial_U\boldsymbol{\Omega})^2\right) + \left(\partial_t\boldsymbol{X}_\parallel\right)^2 - \frac{U^4}{\widetilde{g_{\rm eff}}^2}\left(\partial_U\boldsymbol{X}_\parallel\right)^2\right]. \qquad (19)$$

Introducing the tortoise coordinate σ by

$$\frac{\partial U}{\partial\sigma} = \frac{U^2}{\widetilde{g_{\rm eff}}} \longrightarrow \frac{1}{U} = \frac{\sigma}{\sqrt{\widetilde{g_{\rm eff}}}}, \qquad (20)$$

and also a dimensionless field variable, $\boldsymbol{Y}_\parallel(t,\sigma)$,

$$\boldsymbol{X}_\parallel(t,\sigma) = \frac{\sigma}{\widetilde{g_{\rm eff}}}\boldsymbol{Y}_\parallel(t,\sigma), \qquad (21)$$

one obtains

$$L = \frac{T_{(n,0)}}{2}\int \mathrm{d}\sigma\left[\widetilde{g_{\rm eff}}\left((\partial_t\boldsymbol{\Omega})^2 - (\partial_\sigma\boldsymbol{\Omega})^2\right) + \frac{1}{\widetilde{g_{\rm eff}}}\left(\left(\partial_t\boldsymbol{Y}_\parallel\right)^2 - \left(\partial_\sigma\boldsymbol{Y}_\parallel\right)^2 - \frac{2}{\sigma^2}\boldsymbol{Y}_\parallel^2\right)\right]. \qquad (22)$$

For monochromatic fluctuations $\boldsymbol{\Omega}(\sigma,t) = \boldsymbol{\Omega}(\sigma)\mathrm{e}^{-\mathrm{i}\omega t}$, $\boldsymbol{Y}_\parallel(\sigma,t) = \boldsymbol{Y}_\parallel(\sigma)\mathrm{e}^{-\mathrm{i}\omega t}$, the field equations are reduced to one-dimensional Schrödinger equations

$$-\frac{\partial^2}{\partial\sigma^2}\boldsymbol{\Omega} = \omega^2\boldsymbol{\Omega}, \qquad (23)$$

$$\left(-\frac{\partial^2}{\partial\sigma^2} + \frac{2}{\sigma^2}\right)\boldsymbol{Y}_\parallel = \omega^2\boldsymbol{Y}_\parallel. \qquad (24)$$

One thus finds that the macroscopic Type IIB string hovers around on S^5 essentially via a random walk but, on AdS_5, fluctuations are mostly concentrated on the region $\alpha' U^2 \ll \widetilde{g_{\rm eff}}$, viz. the interior of AdS_5.

3 Strings on D3-brane: Born–Infeld analysis

Let us now turn to the world-volume description of semi-infinite strings ending on D3-branes. From Polchinski's conformal field theory point of view, which is exact at $\lambda_{\rm IIB} = 0$, the end of a fundamental string represents an

[2] This is essentially the same argument as the one due to Callan and Maldacena [11,16]

electric charge (likewise, the end of a D-string represents a magnetic charge). For the semi-infinite string, the electrically charged object has infinite inertia mass, and hence is identified with a heavy quark Q (or anti-quark $\bar{Q}$). An important observation has been advanced recently by Callan and Maldacena [11] (and independently by Gibbons [12] and by Howe, Lambert and West [13]): that the semi-infinite fundamental string can be realized as a deformation of the D3-brane world-volume. It was also emphasized by Callan and Maldacena that a full-fledged Born–Infeld analysis is necessary in order to match the string dynamics correctly.

In this section, we reanalyze the configuration and low-energy dynamics of the semi-infinite strings from the viewpoint of the deformed world-volume of D3-branes. Our ultimate interest being $g_{\rm eff} \to \infty$ and zooming into the anti-de Sitter spacetime, we will proceed our analysis with two different types of Born–Infeld theory. The first is defined by the standard Born–Infeld action, which resums (a subset of) infinite order α' corrections. Since string loop corrections are completely suppressed, results deduced from this are only applicable far away from the D3-branes. As such, we will refer to this regime as being described by *classical* Born–Infeld theory. The second is the conformally invariant Born–Infeld action [4], which resums planar diagrams of ‘t Hooft's large-N expansion in the limit $g_{\rm eff} \to \infty$. With the near-horizon geometry fully taken into account, results obtained from this are directly relevant to the anti-de Sitter spacetime. We will refer to this case as being described by *quantum* Born–Infeld theory.

3.1 Heavy quark in classical Born–Infeld theory

Classical Born–Infeld theory for D3-branes in flat spacetime is described by

$$L_{\rm CBI} = \frac{1}{\lambda_{\rm IIB}} \times \int {\rm d}^3 x \sqrt{\det(\eta_{ab} + \partial_a X_\perp \cdot \partial_b X_\perp + \alpha' F_{ab})}. \tag{25}$$

For a static configuration whose excitation involves only electric and transverse coordinate fields, the Lagrangian is reduced to

$$L_{\rm CBI} \to \frac{1}{\lambda_{\rm IIB}} \int {\rm d}^3 x \Big((1 - \boldsymbol{E}^2)(1 + (\nabla X_\perp)^2) + (\boldsymbol{E} \cdot \nabla X_\perp)^2 - \dot{X}_\perp^2 \Big)^{1/2}. \tag{26}$$

While the equations of motion for $\boldsymbol{E}$ and $\boldsymbol{X}_\perp$ derived from (26) are complicated coupled non-linear equations, for a BPS configuration, the non-linearity simplifies dramatically and reduces to a set of self-dual equations:

$$\nabla X_\perp \cdot \hat{\boldsymbol{\Omega}}_5 = \pm \boldsymbol{E}. \tag{27}$$

Here, $\hat{\boldsymbol{\Omega}}_5$ denotes the angular orientation of the semi-infinite string. The two choices of signs in (27) correspond to quark and anti-quark and are oriented at anti-podal points on $\boldsymbol{\Omega}_5$. Once the above BPS condition (27) is satisfied, the canonical momentum conjugate to the gauge field reduces to the electric field $\boldsymbol{E}$, much as in Maxwell theory. Moreover, such a solution is a BPS configuration. This follows from inserting the relation $\nabla X_i = \pm \boldsymbol{E}$ into the supersymmetry transformation of the gaugino field (in ten-dimensional notation):

$$\begin{aligned} \delta\chi &= \Gamma^{MN} F_{MN} \epsilon \quad (M, N = 0, 1, \cdots, 9) \\ &= \boldsymbol{E} \cdot \Gamma^r \left(\Gamma^0 + \hat{\boldsymbol{\Omega}}_5 \cdot \Gamma \right) \epsilon. \end{aligned} \tag{28}$$

By applying Gauss' law, a semi-infinite strings representing a spherically symmetric heavy quark or an anti-quark of total charge n is easily found[3]:

$$X_\perp \cdot \hat{\boldsymbol{\Omega}}_5 = X_{\perp 0} + \lambda_{\rm IIB} \frac{n}{r} \quad \left(r = |\boldsymbol{x}_\parallel| \right). \tag{29}$$

We emphasize again that the BPS condition is satisfied if all the strings (representing heavy quarks) have the same value of Ω_5 and all the anti-strings (representing heavy anti-quarks) have the anti-podally opposite value of Ω_5.

Now that the heavy quarks and anti-quarks are realized as infinite strings, they can support gapless low-energy excitations. From the D3-brane point of view, these excitations are interpreted as internal excitations on $\mathbf{R}_+ \times S_5$. We would like to analyze these low-energy excitations by expanding the classical Born–Infeld action around a single string configuration. The expansion is tedious but straightforward. Fluctuations to quadratic order come from two sources. The first is from a second-order variation of the transverse coordinates. The second is from the square of the first-order variation involving both transverse coordinates and gauge fields. Evidently, if the background involves non-trivial transverse coordinate fields, this contribution induces mixing between gauge field and transverse coordinate fluctuations. Denoting the gauge field fluctuation as $\mathcal{F}_{\mu\nu}$ and the scalar field fluctuation parallel and perpendicular to the string direction as $Y_\parallel, Y_\perp$, respectively, the low-energy effective Lagrangian is reduced to

$$\begin{aligned} L_{\rm CBI} = \frac{1}{2\lambda_{\rm IIB}} \int {\rm d}^3 x \Big[& (1 + \boldsymbol{E}^2) \mathcal{F}_{0i}^2 - \mathcal{F}_{ij}^2 - 2\boldsymbol{E}^2 \mathcal{F}_{0i} \cdot \partial_i Y_\parallel \\ & + \dot{Y}_\parallel^2 - (1 - \boldsymbol{E}^2)(\partial_i Y_\parallel)^2 \\ & + (1 + \boldsymbol{E}^2) \dot{Y}_\perp^2 - (\partial_i Y_\perp)^2 \Big]. \end{aligned} \tag{30}$$

In order to compare this result with a supergravity analysis, it is necessary to integrate out the world-volume gauge fields. The longitudinal scalar field fluctuation couples only to the electric field. Since the gauge field fluctuations appear through the field strengths, integrating out the gauge field is straightforward. For the S-wave modes, the reduced Lagrangian reads

[3] If all the semi-infinite strings emanate from one of the D3-branes, the center-of-mass factor N should be absent in the expression

$$L_{\rm CBI} = \frac{1}{2\lambda_{\rm IIB}} \int {\rm d}^3 x \Bigg((\partial_t Y_\parallel)^2 - \frac{1}{(1+\boldsymbol{E}^2)} (\partial_r Y_\parallel)^2 + (1+\boldsymbol{E}^2)(\partial_t Y_\perp)^2 - (\partial_r Y_\perp)^2 \Bigg). \quad (31)$$

The structure of this Lagrangian is quite reminiscent of the supergravity fluctuation Lagrangian (10) even though the coordinates involved are quite different. To make a further comparison, we first note that the world-volume coordinate x is *not* the intrinsic coordinate measured *along* the D3-brane world-volume. Since we are studying fluctuations on the D-brane, it is quite important to measure distance using intrinsic D3-brane coordinates. Therefore, we now make a change of variable r to the tortoise coordinate σ:

$$\frac{{\rm d}r}{{\rm d}\tilde{\sigma}} = \frac{1}{\sqrt{\tilde{G}}}; \quad \tilde{G}(r) \equiv (1+\boldsymbol{E}^2) = \left(1 + \frac{n^2 \lambda_{\rm IIB}^2}{r^4}\right). \quad (32)$$

After the change of variables, (31) becomes

$$L_{\rm CBI} = \frac{1}{2\lambda_{\rm IIB}} \int {\rm d}\tilde{\sigma} r^2 \Bigg[\sqrt{\tilde{G}} \left((\partial_t Y_\perp)^2 - (\partial_{\tilde{\sigma}} Y_\perp)^2 \right) + \frac{1}{\sqrt{\tilde{G}}} \left((\partial_t Y_\parallel)^2 - (\partial_{\tilde{\sigma}} Y_\parallel)^2 \right) \Bigg]. \quad (33)$$

Again, the Lagrangian clearly displays the fact that D3-brane coordinate fluctuations parallel and perpendicular to the semi-infinite string propagate at the speed of light even though the string mass density and tension changes spatially. Moreover, the polarization dependence of the string mass density and tension can be understood geometrically from the fact that the proper parallel and orthogonal directions to the D-brane do not coincide with the above fixed background decomposition. In fact, this has been demonstrated explicitly for the case of the open string ending on a D1-brane [17]. Since essentially the same analysis is applicable for the D3-brane, we will not elaborate it further here and move on to the analysis of the boundary conditions.

For a monochromatic transverse fluctuation $Y_\perp(\tilde{\sigma}, t) = Y_\perp(\tilde{\sigma}) {\rm e}^{-{\rm i}\omega t}$, the unitary transformation $Y_\perp \to Y_\perp / r G^{1/4}$ and the change of variables $\tilde{\sigma} \to \tilde{\sigma}/\omega, r \to r/\omega, \lambda_{\rm IIB} \to \lambda_{\rm IIB}/\omega$ yields the fluctuation equation of motion in the form of a one-dimensional Schrödinger equation:

$$\left[-\frac{{\rm d}^2}{{\rm d}\tilde{\sigma}^2} + \tilde{V_\perp}(\tilde{\sigma}) \right] Y_\perp(\tilde{\sigma}) = +1 \cdot Y_\perp(\tilde{\sigma}), \quad (34)$$

where

$$\tilde{V_\perp}(\tilde{\sigma}) = \frac{5\tilde{\epsilon}^{-2}}{(\tilde{\sigma}^2/\tilde{\epsilon}^2 + \tilde{\epsilon}^2/\tilde{\sigma}^2)^3} \quad \left(\tilde{\epsilon} = \sqrt{n \lambda_{\rm IIB}} \omega \right). \quad (35)$$

Note that the functional form of this equation is exactly the same as the one obtained from a supergravity description. Therefore, the fact that the self-adjoint boundary condition of the $Y_\perp$ fluctuation is of Dirichlet type holds in the same way.

Repeating the analysis for monochromatic parallel fluctuations $Y_\parallel(\tilde{\sigma}, t) = Y_\parallel(\tilde{\sigma}) {\rm e}^{-{\rm i}\omega t}$, unitary transformation $Y_\parallel \to r^{-1} G^{1/4} Y_\parallel$ and the same change of variables as above yields the analog one-dimensional Schrödinger equation:

$$\left[-\frac{{\rm d}^2}{{\rm d}\tilde{\sigma}^2} + \tilde{V_\parallel}(\tilde{\sigma}) \right] Y_\parallel(\tilde{\sigma}) = +1 \cdot Y_\parallel(\tilde{\sigma}), \quad (36)$$

where

$$\tilde{V_\parallel}(\tilde{\sigma}) = \frac{(6\tilde{\epsilon}^2/\tilde{r}^2 - \tilde{r}^2/\tilde{\epsilon}^2)}{(\tilde{r}^2/\tilde{\epsilon}^2 + \tilde{\epsilon}^2/\tilde{r}^2)^3}. \quad (37)$$

A comparison to the result (17) shows that, once again, the functional behavior is essentially the same for the supergravity and the classical Born–Infeld side. As such, for low-energy and weak string coupling $g_{\rm IIB} \to 0$, both sides give rise now to Neumann boundary condition, which is another possible self-adjoint extension of the one-dimensional wave operator. Quite surprisingly, we have reproduced Polchinski's boundary condition for an open string ending on D3-branes purely from dynamical considerations, both in spacetime (using a supergravity description) and on the D3-brane world-volume (using the Born–Infeld description).

3.2 Heavy quark in quantum Born–Infeld theory

In the regime $g_{\rm eff} \to \infty$, the D3-brane dynamics is most accurately described by quantum Born–Infeld theory, in which 't Hooft's planar diagrams are resummed over. One immediate question is whether and how the shape and fluctuation dynamics of a semi-infinite string are affected by these quantum corrections. To answer this question, we analyze the semi-infinite string configuration ending on a D3-brane located in the vicinity of $N-1$ other D3-branes. The configuration is depicted in Fig. 1.

The quantum Born–Infeld theory is described by the Lagrangian

$$L_{\rm QBI} = \frac{1}{\lambda_{\rm IIB}} \int {\rm d}^3 x \frac{1}{h} \left[\left(\det \left(\eta_{ab} + h (\partial_a \boldsymbol{X}_\perp \cdot \partial_b \boldsymbol{X}_\perp) + \sqrt{h} F_{ab} \right) \right)^{1/2} - 1 \right],$$

$$h(U) = \frac{g_{\rm eff}^2}{{\rm U}^4} \quad \left(U = |\boldsymbol{X}_\perp| / \ell_s^2 \right).$$

The -1 term inside the bracket originates from the Wess–Zumino term of the D-brane world-volume action and ensures that the ground state has zero energy. For a static world-volume configuration with non-trivial electric and U-fields, one finds

$$L_{\rm QBI} = \frac{1}{\lambda_{\rm IIB}} \int {\rm d}^3 x \frac{1}{h} \Big[\Big(\left(1 - h\boldsymbol{E}^2\right) \left(1 + h (\nabla U)^2\right) + h^2 (\boldsymbol{E} \cdot \nabla U)^2 - h \dot{U}^2 \Big)^{1/2} - 1 \Big]. \quad (38)$$

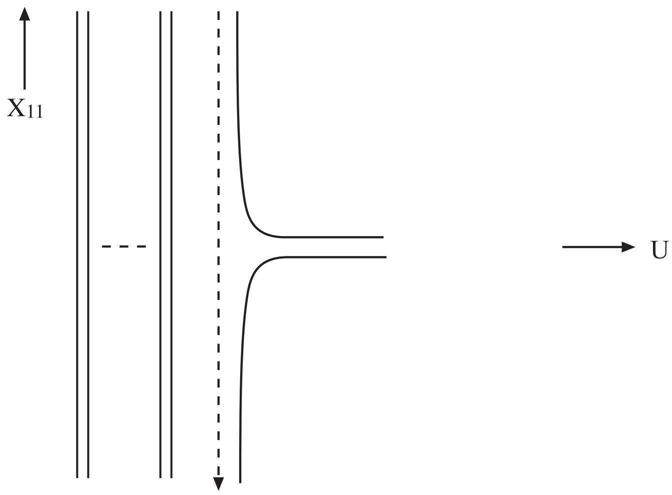

Fig. 1. Macroscopic string as a BPS soliton on a D3-brane world-volume. Large-N corrections induced by branes at $U = 0$ in general give rise to corrections to the shape and low-energy dynamics of the D3-brane

Denoting the quantity inside the square root as L for notational brevity, the canonical conjugate momenta to the gauge field and the Higgs field U are given by

$$\lambda_{\rm IIB}\boldsymbol{\Pi}_A = \frac{1}{L}\left[-\boldsymbol{E}\left(1 + h(\nabla U)^2\right) + h\nabla U(\boldsymbol{E}\cdot\nabla U)\right],$$
$$\lambda_{\rm IIB}\boldsymbol{P}_U = -\frac{1}{L}\dot{U}. \tag{39}$$

We now look for a BPS configuration of the world-volume deformation, as in the case of the classical Born–Infeld theory, that can be interpreted as a semi-infinite string attached to the D3-branes. For a static configuration, the equations of motions read

$$\nabla\cdot\left[\frac{1}{L}(\nabla U(1 - h\boldsymbol{E}^2) + h(\boldsymbol{E}\cdot\nabla U)\boldsymbol{E})\right]$$
$$= \frac{4U^3}{L}h[(\boldsymbol{E}\cdot\nabla U)^2 - \boldsymbol{E}^2(\nabla U)^2], \tag{40}$$
$$\nabla\cdot\left[\frac{1}{hL}(-\boldsymbol{E}(1 + h(\nabla U)^2) + h\nabla U(\boldsymbol{E}\cdot\nabla U))\right] = 0.$$

While coupled in a complicated manner, it is remarkable that the two equations can be solved exactly by the following self-dual BPS equation:

$$\boldsymbol{E} = \pm\nabla U. \tag{41}$$

Remarkably, this self-dual equation is exactly of the same form as the one found for the classical Born–Infeld theory, (20). In this case, $L = 1/h$ and non-linear terms in each equations cancel each other. We emphasize that the Wess–Zumino term -1 in the quantum Born–Infeld Lagrangian, which was present to ensure a vanishing ground-state energy, is absolutely crucial to yield the right-hand side of the first equation of motion, (40). The resulting equation is nothing but the Gauss law constraint, (40):

$$\nabla\cdot\boldsymbol{E} = \nabla^2 U = 0, \tag{42}$$

where the Laplacian is expressed in terms of conformally flat coordinates. A spherically symmetric solution of the Higgs field U is given by

$$U = U_0 + \lambda_{\rm IIB}\frac{n}{r} \quad (r = |\boldsymbol{x}_\parallel|). \tag{43}$$

The interpretation of the solution is exactly the same as in the classical Born–Infeld theory: the gradient of the Higgs field U acts as a source of the world-volume electric field; see (42). From the Type IIB string theory point of view, the source is nothing but n coincident Type IIB fundamental strings attached to the D3-branes. As such, one now has found a consistent world-volume description of the macroscopic Type IIB string in the ‘t Hooft limit.

The total energy now reads

$$E = \int \mathrm{d}^3x\left(\frac{1}{h}[1 + h(\nabla \mathrm{U})^2] - \frac{1}{h}\right)$$
$$= \int \mathrm{d}^3x(\nabla U)^2$$
$$= nU(r = \epsilon). \tag{44}$$

Thus, the total energy diverges with the short-distance cut-off ϵ as in the weak coupling case. Since the above spike soliton is a BPS state and has a non-singular tension the solution remains valid even in the strong coupling regime.

3.3 Quantum Born–Infeld boundary condition

We will now examine the fluctuation of the Born–Infeld fields in the quantum soliton background. The setup is as in the previous subsection – the N multiple D3-branes produce the AdS$_5$ background, and the world-volume dynamics of a single D3-brane in this background is described by the quantum Born–Infeld theory, (38). Keeping harmonic terms, the fluctuation Lagrangian becomes

$$L^{(2)} = -\frac{1}{\lambda_{\rm IIB}}\int \mathrm{d}^3r\frac{1}{2}\left[F^2_{\alpha\beta} - \left(1 + \frac{g^2_{\rm eff}}{\mathrm{U}^4}(\partial_r U)^2\right)F^2_{0\alpha}\right.$$
$$- (\partial_0\chi)^2 + \left(1 - \frac{g^2_{\rm eff}}{\mathrm{U}^4}(\partial_r U)^2\right)(\partial_\alpha\chi)^2$$
$$+ 2\frac{g^2_{\rm eff}}{U^4}(\partial U)^2 F_{0\alpha}\partial_\alpha\chi + 12\frac{U^2}{g^2_{\rm eff}}\chi^2 \tag{45}$$
$$\left. + U^2\left(-\left(1 + \frac{g^2_{\rm eff}}{\mathrm{U}^4}(\partial U)^2\right)(\partial_0\theta)^2 + (\partial_\alpha\theta)^2\right)\right],$$

where χ refers to the radial direction fluctuation, and ψ is the angular fluctuation corresponding to the coordinates θ in the Lagrangian (38). With the Higgs field given as in (43), the above fluctuation Lagrangian is complicated. Thus, we will consider the special situation for which $\mathrm{U}_0 = 0$. In this case, one finds that

$$\frac{g^2_{\rm eff}}{U^4}(\partial_r \mathrm{U})^2 = \frac{g^2_{\rm eff}}{\lambda^2_{\rm IIB}n^2}. \tag{46}$$

This simplifies the fluctuation Lagrangian considerably, yielding

$$L^{(2)} = -\frac{1}{\lambda_{\rm IIB}} \int \mathrm{d}^3 r \frac{1}{2} \left[F_{\alpha\beta}^2 - \left(1 + \frac{g_{\rm eff}^2}{q^2}\right) F_{0\alpha}^2 \right.$$
$$- (\partial_0 \chi)^2 + \left(1 - \frac{Q^2}{q^2}\right)(\partial_\alpha \chi)^2$$
$$+ 2\frac{g_{\rm eff}^2}{\lambda_{\rm IIB}^2 n^2} F_{0\alpha}\partial_\alpha \chi + 12\frac{U^2}{g_{\rm eff}^2}\chi^2$$
$$\left. + U^2 \left(- \left(1 + \frac{g_{\rm eff}^2}{\lambda_{\rm IIB}^2 n^2}\right)(\partial_0\theta)^2 + (\partial_\alpha \theta)^2 \right) \right].$$

One readily finds that the electric field and the radial Higgs field fluctuations are related to each other by

$$\left(1 + \frac{g_{\rm eff}^2}{\lambda_{\rm IIB} n^2}\right) F_{0\alpha} = \frac{g_{\rm eff}^2}{\lambda_{\rm IIB} n^2}\partial_\alpha \chi.$$

Now, integrating out the electric field fluctuation, we find that

$$L^{(2)} = -\frac{1}{\lambda_{\rm IIB}} \Omega_2 \int \mathrm{d}r r^2 \frac{1}{2} \left[F_{\alpha\beta}^2 - (\partial_0 \chi)^2 \right.$$
$$+ \frac{1}{1 + g_{\rm eff}^2/\lambda_{\rm IIB} n^2}(\partial_\alpha \chi)^2 + 12\frac{\lambda_{\rm IIB} n^2}{g_{\rm eff}^2}\frac{1}{r^2}\chi^2$$
$$\left. - \frac{\lambda_{\rm IIB}^2 n^2}{r^2}\left(- \left(1 + \frac{g_{\rm eff}^2}{\lambda_{\rm IIB}^2 n^2}\right)(\partial_0\theta)^2 + (\partial_\alpha \theta)^2 \right) \right],$$
$$(\Omega_2 \equiv \mathrm{Vol}(S_2)).$$

We see that a fluctuation of the magnetic field is non-interacting, and hence we focus on the Higgs field fluctuations only. Make the following change of the radial coordinate and Higgs field[4]:

$$r = \frac{1}{\sqrt{1 + \frac{g_{\rm eff}^2}{\lambda_{\rm IIB}^2 n^2}}}\tilde{r} \quad \text{and} \quad \chi = \lambda_{\rm IIB}\frac{n}{r}\tilde{\chi} = U\tilde{\chi}. \quad (47)$$

The fluctuation Lagrangian then becomes

$$L^{(2)} = \frac{1}{\lambda_{\rm IIB}} \Omega_2 \int \mathrm{d}\tilde{r} \frac{1}{2} q^2 \frac{1}{\sqrt{1 + g_{\rm eff}^2/\lambda_{\rm IIB} n^2}}$$
$$\times \left[(\partial_0 \tilde{\chi})^2 - (\partial_{\tilde{r}} \tilde{\chi})^2 - 12\left(\frac{\lambda_{\rm IIB}^2 n^2}{g_{\rm eff}^2} + 1\right)\frac{\tilde{\chi}^2}{\tilde{r}^2} \right]$$
$$+ \frac{1}{\lambda_{\rm IIB}} \Omega_2 \int \mathrm{d}\tilde{r} \frac{1}{2}\lambda_{\rm IIB}^2 n^2 \sqrt{1 + g_{\rm eff}^2/\lambda_{\rm IIB}^2 n^2}$$
$$\times \left[(\partial_0 \theta)^2 - (\partial_{\tilde{r}} \theta)^2 \right].$$

The overall $\lambda_{\rm IIB}^2 n^2$ factor is actually irrelevant, as it can be eliminated by redefining the θ and $\tilde{\chi}$ fields appropriately.

[4] The change of variable for the χ field renders $\tilde{\chi}$ dimensionless

With an appropriate change of variables as in the supergravity case, we finally obtain the fluctuation equations of motion:

$$\left[-\frac{\partial^2}{\partial_{\tilde{r}}^2} - \omega^2 \right]\theta = 0,$$
$$\left[-\frac{\partial^2}{\partial_{\tilde{r}}^2} + 12\frac{\mathrm{U}^2}{g_{\rm eff}^2} - \omega^2 \right]\tilde{\chi} = 0.$$

Remarkably, while that was not transparent in the intermediate steps, the Higgs field fluctuations turn out to be independent of the $\lambda_{\rm IIB} n$ parameter. This implies that the fluctuations exhibit a universal dynamics, independent of the magnitude of the “quark” charge. The fluctuations comprise essentially the Goldstone modes on S_5 and a harmonically confined radial Higgs field fluctuation localized near $u = 0$. The implications of these characteristics of the fluctuations to the super Yang–Mills theory are discussed elsewhere [20].

3.4 Geometric UV–IR duality

It is remarkable that for both the supergravity and the Born–Infeld theory viewpoints, the fluctuation dynamics is identical given the fact that the σ tortoise coordinate in the supergravity description measures the distance along the $\alpha' U$ direction – a direction *perpendicular* to the D3-brane, while the $\tilde{\sigma}$ tortoise coordinate in the classical Born-infeld description measures the distance *parallel* to the D3-brane, the Yang–Mills distance. The supergravity and the classical Born–Infeld theory provides a dual description of the semi-infinite string as a heavy quark. The reason behind this is that, as α' corrections are taken into account, the D3-brane is pulled by the semi-infinite string and continues deforming until a tensional force balance is achieved. Now that the D3-brane sweeps out in the $\alpha' U$ direction once stretched by charge probes, a balance of tensional force is in order:

$$\frac{1}{R_\parallel} \quad \leftrightarrow \quad \alpha' U, \quad (48)$$

where $R_\parallel = |\boldsymbol{x}_\parallel|$. In particular, the short (long) distance in directions parallel to the D3-brane is related to the long (short) distance in a direction perpendicular to the D3-brane.

We will refer to the “reciprocity relation” (48) as “geometric UV–IR duality” and will derive in later sections the precise functional form of the relation from the consideration of the quark–anti-quark static energy.

4 String–anti-string pair and heavy quark potential

So far, in the previous sections, we have studied the BPS dynamics involving a single probe string. In this section, we extend the study to the non-BPS configuration. We do this again from the Born–Infeld super-Yang–Mills and

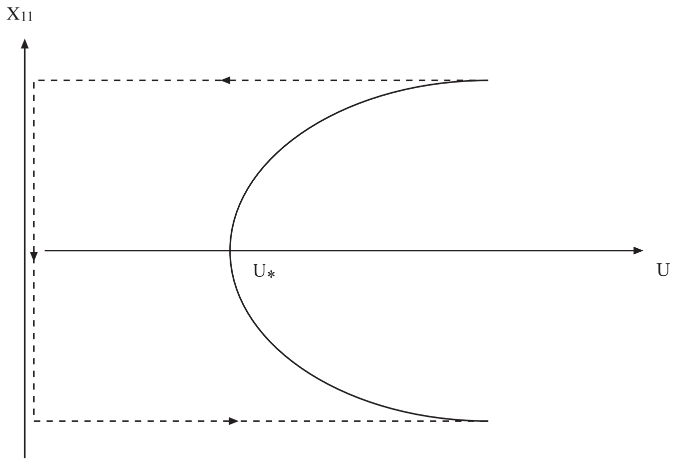

Fig. 2. Non-BPS configuration of a string–anti-string pair as a realization of a heavy quark–anti-quark pair. String corrections smooth out curvature at the two sharp corners

anti-de Sitter supergravity points of view. Among the myriads of non-BPS configurations, the simplest and physically interesting one is a pair of oppositely oriented, semi-infinite strings attached to the D3-brane.

Physically, the above configuration may be engineered as follows. We first prepare a macroscopically large, U-shaped fundamental string, whose tip part is parallel to the D3-brane but the two semi-infinite sides are oriented radially outward; see Fig. 2. As we move this string toward the D3-brane, the tip part will be attracted to the D3-brane and try to form a non-threshold bound-state. The configuration is still not a stable BPS configuration since the two end points from which the semi-infinite sides emanate acts as a pair of *opposite* charges since their Ω_5 orientation is the same. They are nothing but heavy quark–anti-quark pairs. Thus, the two ends will attract each other (since the bound-state energy on the D3-brane is lowered by doing so) and eventually annihilate into radiations. However, in so far as the string is semi-infinite, the configuration will be energetically stable: the inertia of the two open strings is infinite. Stated differently, as the string length represents the vacuum expectation value of the Higgs field, the quark–anti-quark pairs are infinitely heavy. In this way, we have engineered a static configuration of a $(Q\overline{Q})$ pair on the D3-brane.

The $(Q\overline{Q})$ configuration is of some interest since it may tell us whether the $d=4, \mathcal{N}=4$ super-Yang–Mills theory exhibits confinement. The theory has a vanishing β function, and hence no dimensionally transmuted mass gap either. Therefore, one might be skeptical to the generation of a physical scale from a *gedanken* experiment using the above configuration. The result we will get is not in contradiction, however, as the scale interpreted as a sort of "confinement" scale is really residing in the AdS_5 spacetime. It is a direct consequence of spontaneously broken conformal invariance of the super-Yang–Mills theory. Therefore, the "confinement" behavior in the AdS_5 spacetime ought to be viewed as "Coulomb" behavior in super-Yang–Mills theory. Once again, the interpretation relies on the earlier observation that parallel and perpendicular directions to the D3-brane are geometrically dual to each other.

4.1 Quark–anti-quark pair: String in anti-de Sitter space

We first construct the aforementioned string configuration corresponding to $Q\overline{Q}$ pair on the D3-brane from anti-de Sitter supergravity. To find the configuration we find it most convenient to study portions of the string separately. Each of the two semi-infinite portions is exactly the same as a single semi-infinite string studied in the previous section. Thus, we concentrate mainly on the tip portion that is about to bound to the D3-brane. The portion cannot be bound entirely parallel to the D3-brane since it will cause a large bending energy near the location we may associate with Q and $\overline{Q}$. The minimum energy configuration would be literally like an U-shape. We now show that this is indeed what happens.

We now repeat the analysis of a test string in a supergravity background of N D3-branes. For a static configuration, the Nambu–Goto Lagrangian is exactly the same as (7):

$$L_{\mathrm{NG}} \to \int \mathrm{d}\sigma \sqrt{\boldsymbol{X}'^{\,2}_{\perp} + \frac{1}{G}\boldsymbol{X}'^{\,2}_{\parallel}}. \tag{49}$$

From the equations of motion, we find that the other possible solution is when the string is oriented parallel to the D3-brane. This yields precisely the static gauge configuration

$$X^0 = t = \tau, \quad \boldsymbol{X}_{\parallel} = \sigma \hat{\boldsymbol{n}}. \tag{50}$$

Then, the two equations of motion (8) become

$$\left(\frac{\boldsymbol{X}'_{\perp}}{\sqrt{\boldsymbol{X}'^{\,2}_{\perp} + G^{-1}}}\right)' = (\nabla_{\boldsymbol{x}_{\perp}} G^{-1}),$$
$$\left(\frac{G^{-1}}{\sqrt{\boldsymbol{X}'^{\,2}_{\perp} + G^{-1}}}\right)' = 0. \tag{51}$$

We now consider the non-BPS $Q\overline{Q}$ configuration studied earlier. Since the two semi-infinite strings are oriented parallel on Ω_5 we only consider an excitation of the $\alpha' U$ coordinate. From the equation of motion, the first of (51),

$$-\frac{1}{G}U'' + \frac{1}{2}\left(\partial_{\mathrm{U}}\frac{1}{G}\right)\left(2U'^2 + \frac{1}{G}\right) = 0, \tag{52}$$

one can obtain the first integral of motion:

$$G^2 U'^2 + G = \frac{g^2_{\mathrm{eff}}}{U^4_*}, \tag{53}$$

where we have chosen a convenient parameterization of the integration constant. This is in fact the same as the other conserved integral, the second of (51) and shows that the equations are self-consistent.

Denoting $Z \equiv U_*/U$, the solution to (53) can be found in an implicit functional form:

$$(x_\| - d/2) = \pm \frac{g_{\text{eff}}}{U_*} \left[\sqrt{2} E \left(\arccos Z, \frac{1}{\sqrt{2}} \right) - \frac{1}{\sqrt{2}} F \left(\arccos Z, 1/\sqrt{2} \right) \right]. \quad (54)$$

Here, $F(\phi, k), E(\phi, k)$ denote the elliptic integrals of the first and second kinds. It is easy to visualize that the solution describes a monotonic lifting of the U direction fluctuation (thus away from the D3-brane plane) and diverges at finite distance along $x_\|$. For our choice, they are at $x_\| = 0$ and d. This prompts us to interpret the integration constant d in (54) as the separation between quark and anti-quark, measured in $x_\|$ coordinates. The string is bent (roughly in U-shape) symmetrically about $x_\| = d/2$. As such, the inter-quark distance measured *along* the string is not exactly the same as d. The proper distance along the string is measured by the U coordinate. The relation between the coordinate separation and proper separation is obtained easily by integrating over the above (49). It yields

$$\frac{d}{2} = \frac{g_{\text{eff}}}{\text{U}_*} \left[\sqrt{2} E \left(\frac{\pi}{2}, \frac{1}{\sqrt{2}} \right) - \frac{1}{\sqrt{2}} F \left(\frac{\pi}{2}, \frac{1}{\sqrt{2}} \right) \right] \quad (55)$$
$$= \frac{g_{\text{eff}}}{U_*} C_1 \quad \left(C_1 = \sqrt{\pi} \Gamma(3/4)/\Gamma(1/4) = 0.59907... \right).$$

This formula implies that the integration constant U_* would be interpreted as the height of the U-shaped tip along the U coordinate. Up to numerical factors, the relation again exhibits the "geometric duality" (32) between the Yang–Mills coordinate distance d and the proper distance U_*.

Using the first integral of motion, the inter-quark potential is obtained straightforwardly from the Born–Infeld Lagrangian. The proper length of the string is infinite, so we would expect a linearly divergent (in the U coordinate) energy. Thus, we first calculate the regularized expression of the energy by excising out a small neighborhood around $x_\| = 0, d$:

$$V_{Q\overline{Q}}(d) = \lim_{\epsilon \to 0} n \left[\sqrt{G_*} \int_0^{d/2-\epsilon} \mathrm{d}x_\| G^{-1} \right] \quad (56)$$
$$= \lim_{U \to \infty} n \left[2U_* \int_1^U \mathrm{d}t \frac{t^2}{\sqrt{t^4 - 1}} \right]$$
$$= 2nU_* \left[U + \frac{1}{\sqrt{2}} K(1/\sqrt{2}) - \sqrt{2} E(1/\sqrt{2}) + \mathcal{O}(U^{-3}) \right].$$

The last expression clearly exhibits the infinite energy originating from the semi-infinite strings and indeed it is proportional to the proper length $2U$. After subtracting (or renormalizing) the string self-energy, the remaining, finite part may now be interpreted as the inter-quark potential. An amusing fact is that it is proportional to the inter-quark distance when measured in the U coordinate. One might be tempted to interpret the inter-quark potential as being in fact a Coulomb potential by using the relation (51). However, it does *not* have the expected dependence on the electric charges: instead of a quadratic dependence, it only grows linearly. Because of this, we suspect that the interpretation of the static $Q\overline{Q}$ potential is more natural when viewed as a linearly confining potential in the U direction in AdS_5.

The static inter-quark potential shows several peculiarities. First, the potential is purely Coulombic, viz. inversely proportional to the separation distance. This, however, is due to the underlying conformal invariance. Indeed, at the critical point of the second-order phase transition (where conformal invariance is present), it was known that the Coulomb potential is the only possible behavior [21]. Second, most significantly, the static quark potential strength is *non-analytic* in the effective 't Hooft coupling constant, g^2_{eff}. The quark potential is an experimentally verifiable physical quantity, and, in the weak 't Hooft coupling domain, it is well known that physical quantities ought to be analytic in g^2_{eff}, at least, within a finite radius of convergence around the origin. Moreover, for $d = 4$, $\mathcal{N} = 4$ super-Yang–Mills theory, we do not expect a phase transition as the 't Hooft coupling parameter is varied. Taking the aforementioned non-analyticity of the square-root branch cut type as a prediction for the strongly coupled super-Yang–Mills theory, we conjecture that there ought to be two distinct strong coupling systems connected smoothly to one and the same weakly coupled super-Yang–Mills theory. To what extent these two distinct systems are encoded into a single AdS_5 supergravity is unclear, and hence poses an outstanding issue to be resolved in the future.

4.2 Heavy quark–anti-quark pair: Quantum Born–Infeld analysis

Let us begin with a quantum Born–Infeld analysis of the heavy quark–anti-quark pair. In earlier sections, we have elaborated that quarks and anti-quarks correspond to semi-infinite strings of opposite Ω_5 orientation angle. That this is a BPS configuration can be understood in several different ways. Consider a string piercing the D3-brane radially. The simplest is from the gaugino supersymmetry transformation, (28). Residual supersymmetry is consistent among individual semi-infinite strings if and only if their Ω_5 angular orientations are all the same for the same charges and anti-podally opposite for opposite charges. Alternatively, at the intersection locus, one can split the string and slide the two ends in opposite directions. This does not cost any energy since the attractive electric force is balanced by a repulsive $\alpha' U$ gradient force. This BPS splitting naturally gives rise to a quark–anti-quark configuration in which semi-infinite strings are anti-podally opposite on Ω_5.

The fact that $Q\overline{Q}$ does not exert any force in this case is not a contradiction at all. The Coulomb force between Q and $\overline{Q}$ is cancelled by a gradient force of the $\alpha' U$ field. This already indicates that we have to be careful in interpreting

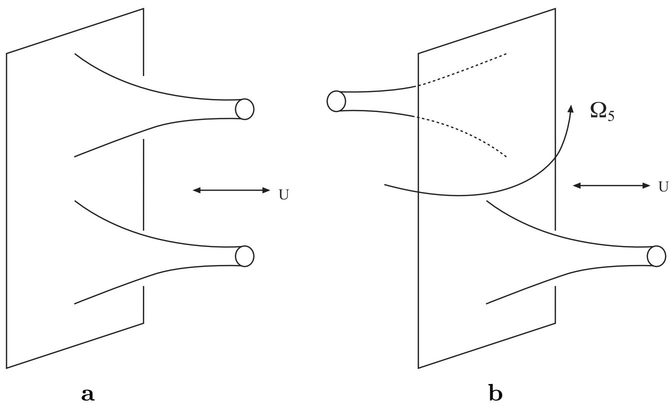

Fig. 3a,b. Heavy ($Q\overline{Q}$) realization via deformation of the D3-brane world-volume. A highly non-BPS configuration **a** corresponds to two throats located at the same point on Ω_5. For the BPS configuration **b**, two throats are at anti-podal points on Ω_5. By continuous rotation on Ω_5, **b** can be turned into **a** and vice versa

the evolution of $Q\overline{Q}$ on the D3-brane as a timelike Wilson loop of the four-dimensional gauge fields only. The more relevant quantity is the full ten-dimensional Wilson loop:

$$W[C] = \exp[\mathrm{i} \oint (A_\alpha \mathrm{d}x^\alpha + \dot{\boldsymbol{X}}_\perp \cdot \mathrm{d}x_\perp)]. \tag{57}$$

From the BPS point of view, it simply states that, for example, in evaluating a static potential between heavy quark and anti-quark, one has to include *all* long-range fields that will produce the potential.

A little thought concerning the BPS condition (41) indicates that there is yet another configuration that may be interpreted as a static $Q\overline{Q}$ state. If we take a semi-infinite string representing a quark with the positive sign choice in (37) and superimpose another semi-infinite string representing an anti-quark with the negative sign choice, then we obtain a $Q\overline{Q}$ configuration in which the Ω_5 angular positions are the *same*. In this case, it is easy to convince oneself that both the Coulomb force and the U-field gradient force are attractive, and hence produce a non-trivial $Q\overline{Q}$ static potential. Indeed, starting from the BPS $Q\overline{Q}$ configuration with opposite Ω_5 orientations mentioned just above, one can deform it into the present non-BPS $Q\overline{Q}$ configuration by rotating one of the semi-infinite string on Ω_5 relative to the other; see Fig. 3 for illustration. It should also be clear that it is the gradient force of scalar fields in the transverse directions that changes continuously as the relative Ω_5 angle is varied.

While an explicit solution describing the $Q\overline{Q}$ configuration might be possible, we were not able to find the solution in any closed form starting from the quantum Born–Infeld action. Therefore, in this section, we will calculate the static potential for the non-BPS $Q\overline{Q}$ configuration with an asymptotic approximation. Namely, if the separation between the semi-infinite string representing a quark and another representing an anti-quark is wide enough, the field configuration may be approximated to a good degree by a linear superposition of two pairs of single string BPS solutions with opposite sign choice in (43). For the $\alpha' U$ field, the approximate configuration is given by

$$U(\boldsymbol{r}) := U_0 + n\lambda_{\mathrm{IIB}} \left(\frac{1}{|\boldsymbol{r} + \boldsymbol{d}/2|} + \frac{1}{|\boldsymbol{r} - \boldsymbol{d}/2|} \right), \tag{58}$$

while the electric field is a linear superposition of differences of the gradients of each term in (58). Note that the inflection point of the $\alpha' U$ field is around the midpoint $\boldsymbol{r} = 0$ between Q and $\overline{Q}$. If we denote the lift of the U-field at this point as U_*, measured relative to the asymptotic one U_0, it is given by

$$U_* \approx \frac{4n}{|\boldsymbol{d}|}. \tag{59}$$

Interestingly, a short-distance limit (i.e. an inter-quark separation $|\boldsymbol{d}| \to 0$) in the gauge theory corresponds to a long-distance limit ($U_* \to \infty$) in anti-de Sitter supergravity and vice versa.

Let us now estimate the static $Q\overline{Q}$ potential. If we insert the linear superposition of solutions to the energy functional, (40), there are self-energy contributions of the form precisely as in the last line in (40). Subtracting (or rather renormalizing) these self-energies, we are left with the interaction energy

$$V(d) \sim 2n^2 \int \mathrm{d}^3 x \frac{1}{|\boldsymbol{r} + \boldsymbol{d}/2|^2} \frac{1}{|\boldsymbol{r} - \boldsymbol{d}/2|^2} \times (\hat{\boldsymbol{r}} + \hat{\boldsymbol{d}}/2) \cdot (\hat{\boldsymbol{r}} - \hat{\boldsymbol{d}}/2). \tag{60}$$

The integral is finite and, by dimensional analysis, is equal to

$$V_{Q\overline{Q}}(d) \sim 2n^2 \frac{C_{\mathrm{BI}}}{|\boldsymbol{d}|}, \tag{61}$$

where the coefficient C_2 depends on g_{st} and N. The dimensionless numerical coefficient C_{BI}, which depends critically on λ_{IIB} and N through the relation (43), can be calculated, for example, by the Feynman parameterization method. The interaction potential is indeed a Coulomb potential – inversely proportional to the separation and proportional to the charge-squared. Utilizing the "geometric duality" relation (59), it is also possible to re-express the static potential by

$$V_{Q\overline{Q}}(U_*) \sim \frac{1}{2} n U_* C_{\mathrm{BI}}. \tag{62}$$

Recall that U_* was a characteristic measure of the $\alpha' U$ field lift relative to the asymptotic value U_0 (See Fig. 3). Since this is caused by bringing in Q and $\overline{Q}$ of the *same* orientation, the interpretation would be that the static $Q\overline{Q}$ potential is produced by a U_* portion of the string due to the presence of a neighbor non-BPS string. In some sense, the $Q\overline{Q}$ pair experiences a confining force in the $\alpha' U$ direction. The fact that (62) is proportional *linearly* to the charge n is another hint to this "dual" interpretation. The result (62), however, does not expose the aforementioned non-analyticity of the square-root branch cut type in the previous subsection. We interpret this provisionally as the assertion that the Born–Infeld theory is insufficient for a full-fledged description of the strong coupling dynamics.

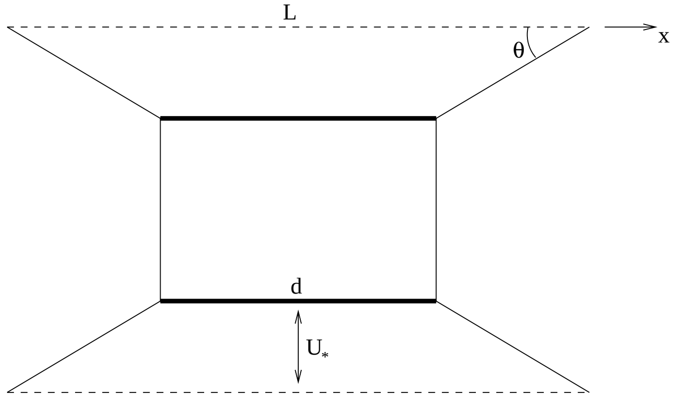

Fig. 4. Non-BPS configuration of quark–anti-quark pair on a D-string

Now that we have found two distinct $Q\overline{Q}$ configurations, we can estimate the $Q\overline{Q}$ static potential purely due to the Coulomb interaction. Recall that, for the BPS $Q\overline{Q}$ configuration, the Coulomb interaction energy was cancelled by the $\alpha' U$ field gradient energy. On the other hand, for the non-BPS $Q\overline{Q}$ configuration, the two add up. Thus, by taking an average of the two, we estimate that the purely Coulomb potential between the static $Q\overline{Q}$ equals half of (61) or, equivalently, of (62).

4.3 Heavy quark potential in one dimension

In the previous subsection, we have estimated the $Q\overline{Q}$ static potential only approximately by linearly superimposing two opposite sign BPS string configurations. To ascertain that this is a reasonable approximation, we study a simpler but exactly soluble example of the $Q\overline{Q}$ potential: a pair of oppositely oriented fundamental strings hung over two parallel, widely separated D-strings.

Consider, as depicted in Fig. 4, a pair of D-strings of length L along the x-directon, whose ends are at fixed position. The two fundamental strings of opposite orientation are connected to the two D-strings and are separated by a distance d in the x direction. At $\lambda_{\rm IIB} \to 0$, the fundamental strings obey the Polchinski string boundary conditions and are freely sliding on the D-string.

Once $\lambda_{\rm st}$ is turned on, the string network gets deformed into a new equilibrium configuration. It is intuitively clear what will happen: the two fundamental strings will attract the two D-strings. In doing so, the length of the fundamental strings is shortened. Since the two funadamental strings are oppositely oriented, they will attract each other and eventually annihilate. In the weak coupling regime, however, the force is weak compared to the inertial mass of the fundamental string. We shall calculate the potential between them in this weak coupling regime.

This energy difference is given by

$$V_{Q\overline{Q}}(d) = {\rm d}\left[\sqrt{\frac{1}{\lambda_{\rm IIB}^2} + n^2} - \frac{1}{\lambda_{\rm IIB}}\right]$$

$$\approx {\rm d}\left[\frac{1}{2}n^2\lambda_{\rm IIB}\right]. \quad (63)$$

This indeed represents the static $(Q\overline{Q})$ potential. As expected for the Coulomb interaction, the energy is proportional to the quark's charge-squared n^2. It is also proportional to the string coupling $\lambda_{\rm st}$, which is also proportional to $g_{\rm YM}^2$.

The potential can be interpreted differently. The four portions of D-strings between each string junctions and the fixed ends are now all bent by the same angle θ relative to the x-axis. From the requirement of tensional force balance at each string triple junction one finds easily that

$$\tan\theta = n\lambda_{\rm IIB}. \quad (64)$$

Then, a simple geometric consideration leads to the relation that the shortening of the fundamental string denoted by U_* is given by

$$2U_* = (L-d)\tan\theta. \quad (65)$$

Using these relations for (63) we now find that

$$V_{Q\overline{Q}}(U_*) = nU_* \quad (66)$$

plus an irrelevant bulk contribution. In this alternative form, it is clear that the static potential energy originates from the deformation of the string network, which in turn reduces the length of the fundamental strings.

Note that in deriving the above results, we have linearly superposed two triple string junctions, each satisfying BPS conditions $E = \pm\nabla_x U$ respectively. The linearly superposed configuration then breaks the supersymmetries completely. Nevertheless, at weak coupling and for macroscopically large size, we were able to treat the whole problem quasi-statically, thanks to the (almost) infinite inertial mass of the fundamental strings. Thus, approximations and results are exactly the same as for $(Q\overline{Q})$ on D3-branes.

4.4 θ-dependence of inter-quark potential

The $d = 4$, $\mathcal{N} = 4$ super-Yang–Mills theory contains two coupling parameters $g_{\rm YM}^2$ and θ, the latter being a coefficient of ${\rm Tr}(\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta})/32\pi^2$. From the underlying Type IIB string theory, they arise from the string coupling parameter $\lambda_{\rm st}$ and Ramond–Ramond zero-form potential C_0. They combine into a holomorphic coupling parameter

$$\tau = \frac{\theta}{2\pi} + {\rm i}\frac{4\pi}{g_{\rm YM}^2 N}$$
$$= C_0 + {\rm i}\frac{1}{\lambda_{\rm IIB}}. \quad (67)$$

From the gauge theory point of view, one of the interesting questions is the θ-dependence of the static quark potential. Under $d = 4$ P and CP, the former is odd while the latter is even. Thus, the static quark potential should be an even function of θ. The θ range is $(0, 2\pi)$. Then, the periodicity of θ (i.e. the T-transformation of $SL(2,\mathbf{Z})$ and invariance of the static quark potential under the parity transformation immediately dictate that the quark potential should be symmetric under $\theta \to -\theta$ and $\pi-\theta \to \pi+\theta$. This yields a cuspy form for the potential. Since the whole physics descends from the $SL(2,\mathbf{Z})$ S-duality, let us make a little calculation in a closely related system: the triple junction network of (p,q) strings. This system will exhibit most clearly the very fact that string tension is reduced most at $\theta=\pi$. That this is so can be seen from replacing n in the previous analysis by a θ-angle rotated dyon case:

$$n \to \sqrt{(n-\theta m)^2 + \frac{m^2}{\lambda_{\rm IIB}^2}}. \qquad (68)$$

The whole underlying physics can be understood much clearer from the D-string junctions. Consider a $(0,1)$ D-string in the background of a Ramond–Ramond zero-form potential. The Born–Infeld Lagrangian reads

$$L_{\rm D1} = \frac{T}{\lambda_{\rm IIB}} \int {\rm d}x \sqrt{1+(\nabla X)^2 - F^2} + C_0 \wedge F. \qquad (69)$$

Consider a $(1,0)$ fundamental string attached to a D-string at the location $x=0$. The static configuration of the triple string junction is then found by solving the equation of motion. In the $A_1=0$ gauge,

$$\nabla \left(\frac{-\nabla A_0}{\sqrt{1+(\nabla X_9)^2 - (\nabla A_0)^2}} - \lambda_{\rm IIB} C_0 \right) = \lambda_{\rm IIB}\delta(x). \qquad (70)$$

The solution is $X_9 = a^{1/2} A_0$ for the continuous parameter a, where

$$\frac{\nabla A_0}{\sqrt{1-(1-a)(\nabla A_0)^2}} = \lambda_{\rm st}\theta(x_1) - \lambda_{\rm IIB} C_0. \qquad (71)$$

Substituting the solution to the Born–Infeld Lagrangian, we find the string tension of the D-string:

$$T_{\rm D} = \begin{cases} \sqrt{\dfrac{1}{\lambda_{\rm IIB}^2} + (1-C_0)}, & x_1 > 0, \\ \sqrt{\dfrac{1}{\lambda_{\rm IIB}^2} + C_0^2}, & x_1 < 0. \end{cases}$$

Clearly, the tension of the $(1,1)$ string (on which the electric field is turned on) attains the minimum when $C_0 = 1/2$, viz. $\theta=\pi$. Moreover, in this case, the D-string bends symmetrically around the junction point $x_1=0$, reflecting the fact that P and CP symmetries are restored at $\theta=\pi$.

5 Further considerations

In this section, we take up further the present results and speculate on two issues that might be worthy of further study.

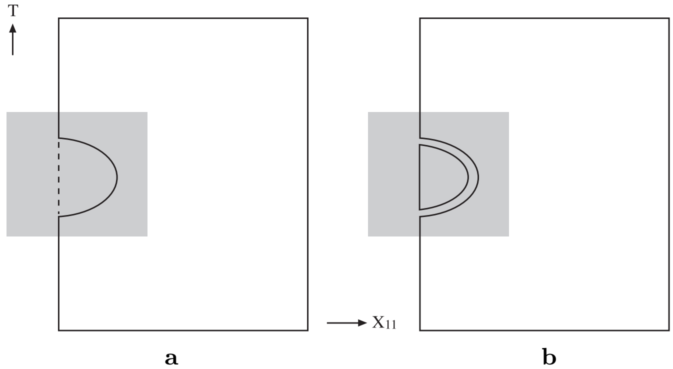

Fig. 5a,b. Conformal transformation causes local recoil of a timelike loop. Back tracking at large N is equivalent to a pair creation process

5.1 Dynamical realization of large-N loop equation

It is well known that the Wilson loop

$$W[X] = \exp \oint_C {\rm d}s \dot{X}^M A_M(X(s)) \qquad (72)$$

satisfies the classical identity

$$\int_0^\epsilon {\rm d}\sigma \frac{\delta^2}{\delta X_M(+\epsilon)\delta X_M(\epsilon)} W[X] = \nabla^M F_{MN}(X(0)) \dot{X}^N(0) W[X]. \qquad (73)$$

Physically, this equation can be interpreted as a variation of the Wilson loop as the area enclosed is slightly deformed.

More recently, based on a dual description of large-N gauge theory in $1+1$ dimensions in terms of near-critical electric field on a D-string, Verlinde [24] has shown that the Wilson loop equation follows as the conformal Ward identity on the string world-sheet. An immediate question that arises is: relying on the $SO(4,2)$ conformal invariance of large-N super-Yang–Mills gauge theory, can one extend Verlinde's result and derive the large-N loop equation? In what follows, we would like to present rather heuristic arguments why and how conformal invariance might play some role in this direction.

Classically the large-N loop equation asserts invariance of the Wilson loop average under a small variation of the area enclosed by the loop. Let us now restrict ourselves to timelike Wilson loops and apply a small deformation of the contour C. As the contour C of the timelike Wilson loop represents a straight world-line of a heavy quark–anti-quark pair, the adiabatic local deformation of the Wilson loop may be interpreted as a result of the acceleration of an initially static quark Q and subsequent deceleration back to the original static quark world-line during a small time interval. This is depicted in Fig. 5a. Normally, such acceleration and deceleration requires turning on and off some adiabatic electric field in the region near the quark Q trajectory (the shaded region of Fig. 5a).

However, for conformally invariant Yang–Mills theory, there is an amusing possibility that the accelerating (decelerating) charge configuration can be achieved via conformal transformation *without* a background electric field. Recall that, in Lorentz invariant theory, it is always possible that a static configuration can be brought into a uniformly boosted configuration by the application of a Lorentz transformation. What conformally invariant theory does is more than that; it can even relate, for example, a uniformly accelerated (decelerated) configuration by a conformal transformation to a static (or uniformly boosted) configuration. Indeed, if we apply a special $SO(4,2)$ conformal transformation of an inversion with a translation by a^μ followed by another inversion, we have

$$x^\mu \to x^{\mu'} = \frac{x^\mu + a^\mu x^2}{1 + 2a \cdot x + a^2 x^2}. \tag{74}$$

If we set $a^\mu = (0, -(1/2)\boldsymbol{a})$, we obtain

$$t' = \frac{t}{1 - \boldsymbol{z} \cdot \boldsymbol{x} + \frac{1}{4}\boldsymbol{a}^2(\boldsymbol{x}^2 - t^2)},$$
$$\boldsymbol{x}' = \frac{\boldsymbol{x} + \frac{1}{2}\boldsymbol{z}(t^2 - \boldsymbol{x}^2)}{1 - \boldsymbol{a} \cdot \boldsymbol{x} + \frac{1}{4}\boldsymbol{z}^2(\boldsymbol{x}^2 - t^2)}. \tag{75}$$

Thus, the original trajectory of the static configuration at $\boldsymbol{x} = 0$ is now transformed into

$$t_* = \frac{t}{1 - \frac{1}{4}\boldsymbol{a}^2 t^2}, \quad \boldsymbol{x}_* = \frac{\frac{1}{2}\boldsymbol{a}t^2}{1 - \frac{1}{4}\boldsymbol{a}^2 t^2}, \tag{76}$$

which, for $|t| < 2/|\boldsymbol{a}|$, represents the coordinates of a configuration with constant acceleration $\boldsymbol{a}$ passing through the origin $\boldsymbol{x}_* = 0$ at $t_* = 0$.

Thus, if one performs instantaneous special conformal transformations on a finite interval along the heavy quark Q trajectory, then it would indeed be possible to show that a timelike Wilson loop is equivalent to a deformed Wilson loop (by the conformal transformation, however, only timelike deformations can be realized). Since the antipodally oriented $Q\overline{Q}$ pair is a BPS state, it might even be possible to generate a four-quark (of which two are virtual BPS states) intermediate state by a variant of the conformation transformation, as depicted in Fig. 5b. Details of this issue will be reported elsewhere [23].

5.2 Multi-prong strings

Moving a step further, can we manufacture a static configuration that may be an analog of the *baryon* in QCD out of Type IIB strings? For the gauge group $SU(N)$, the baryon is a gauge singlet configuration obeying N-ality. Clearly, we need to look for a string configuration that can be interpreted as a N-quark state on the D3-brane worldvolume. Recently, utilizing a triple BPS string junction [17,25], such a configuration has been identified [26]: an N-pronged string junction interconnecting N D3-branes.

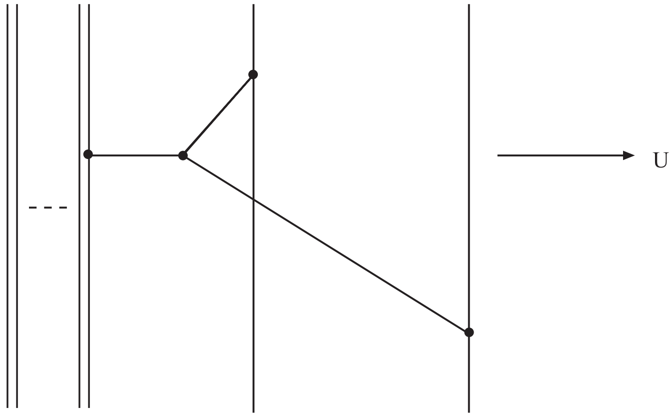

Fig. 6. Macroscopic string as a BPS soliton on the D3-brane world-volume. Large-N corrections induced by branes at $U = 0$ in general give rise to corrections to the shape and low-energy dynamics of the D3-brane

For example, for the gauge group $SU(3)$ realized by three D3-branes, the multi-monopole configuration that may be interpreted as the static baryon is a triple string junction as depicted in Fig. 6. The N-pronged string junction is a natural generalization of this, as can be checked from counting of the multi-monopole states and comparison with the (p,q) charges of the Type IIB string theory.

The N-pronged string junction also exhibits the dynamics of marginal stability as we move around the D3-branes to which all prongs are attached [26]. Adapted to the present context, for example, in the situation in Fig. 6, this implies that as the triple junction point is moved around by moving the position of the two outward D3-branes as well as their Ω_5 angular coordinates, the triple string junction will decay once the inner prong becomes shorter below the curve of marginal stability. The final configuration is easily seen to be a pair of macroscopic strings, each one connecting to the two outer D3-branes separately.

5.3 Quarks and $(Q\overline{Q})$ at finite-temperature

So far, our focus has been, via the AdS–CFT correspondence, the holographic description of strongly coupled $N = 4$ super-Yang–Mills theory at zero temperature. The AdS–CFT correspondence, however, is not only for the super-Yang–Mills theory at zero temperature, but also is extendible for the theory at finite temperature. Is it then possible to understand the finite-temperature physics of quark dynamics and static quark potential, again, from the AdS–CFT correspondence? We will relegate the detailed analysis to a separate work [20], and, in this subsection, summarize what is known from the super-Yang–Mills theory side and propose the set-up for a holographic description.

At a finite critical temperature $T = T_c$, pure $SU(N)$ gauge theory exhibits a deconfinement phase transition. The relevant order parameter is the Wilson–Polyakov loop:

$$P(\boldsymbol{x}) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}\int_0^{\frac{1}{T}} A_0(\boldsymbol{x})\mathrm{d}t\right). \tag{77}$$

Below the critical temperature $T < T_\mathrm{c}$, $\langle P\rangle = 0$, and QCD confines. Above $T > T_\mathrm{c}$, $\langle P\rangle$ is non-zero and takes values in $\mathbf{Z}_N$, the center group of $SU(N)$. Likewise, the two-point correlation of parallel Wilson–Polyakov loops,

$$\begin{aligned}\Gamma(\boldsymbol{d}, T) &\equiv \langle P^\dagger(\mathbf{0})P(\boldsymbol{d})\rangle_T \\ &= \mathrm{e}^{-\mathcal{F}(\boldsymbol{d},T)/T} \approx \mathrm{e}^{-V_{\mathrm{Q}\overline{\mathrm{Q}}}(\boldsymbol{d},T)/T},\end{aligned} \tag{78}$$

measures the static potential at finite temperature between quark and anti-quark separated by a distance d.

At sufficiently high temperature, thermal excitations produce a plasma of quarks and gluons and give rise to a Debye mass $m_\mathrm{E} \approx g_\mathrm{eff}T$ (which is responsible for screening color electric flux) and magnetic mass $m_\mathrm{M} \approx g^2_\mathrm{eff}T$ (which corresponds to the glueball mass gap in the confining three-dimensional pure gauge theory). Their effects are captured by the asymptotic behavior of the heavy quark potential:

$$\begin{aligned}V_{\mathrm{Q}\overline{\mathrm{Q}}}(\boldsymbol{d}, T) \approx &-C_\mathrm{E}\tfrac{1}{|\boldsymbol{d}|^2}\mathrm{e}^{-2m_\mathrm{E}|\boldsymbol{d}|} + \cdots \quad && C_\mathrm{E} = \mathcal{O}(g^4_\mathrm{YM}), \\ &-C_\mathrm{M}\tfrac{1}{|\boldsymbol{d}|}\mathrm{e}^{-m_\mathrm{M}|\boldsymbol{d}|} + \cdots \quad && C_\mathrm{M} = \mathcal{O}(g^{12}_\mathrm{YM}).\end{aligned} \tag{79}$$

It is known that at finite temperature the large-N and strong coupling limit of $d = 4$, $\mathcal{N} = 4$ supersymmetric gauge theory is dual to the near-horizon geometry of near-extremal D3-branes in Type IIB string theory. The latter is given by a Schwarzschild–anti-de Sitter Type IIB supergravity compactification:

$$\begin{aligned}\mathrm{d}s^2 = \alpha'\Bigg[&\frac{1}{\sqrt{G}}\left(-H\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2_\parallel\right) \\ &+\sqrt{G}\left(\frac{1}{H}\mathrm{d}U^2 + U^2\mathrm{d}\boldsymbol{\Omega}^2_5\right)\Bigg],\end{aligned} \tag{80}$$

where

$$\begin{aligned}G &\equiv \frac{g^2_\mathrm{eff}}{U^4}, \\ H &\equiv 1 - \frac{U_0^4}{U^4} \quad \left(U_0^4 = \frac{2^7\pi^4}{3}g^4_\mathrm{eff}\frac{\mu}{N^2}\right).\end{aligned} \tag{81}$$

The parameter μ is interpreted as the free energy density on the near-extremal D3-brane; hence, $\mu = (4\pi^2/45)N^2T^4$. In the field theory limit $\alpha' \to 0$, μ remains finite. In turn, the proper energy $E_\mathrm{sugra} = (g_\mathrm{eff}/\alpha')^{1/2}\mu/U$ and the dual description in terms of modes propagating in the above supergravity background is expected to be a good approximation.

Hence, the question is whether the Debye screening of the static quark potential (79), or any strong coupling modification thereof, can be understood from the holographic description in the background (80). In [20], we were able to reproduce a result qualitatively in agreement with (79). The strong coupling effect again shows up through the non-analytic dependence of the potential to the ‘t Hooft coupling parameter, exactly the same as for the zero-temperature static potential. In [28], we have also found a result indicating that the finite-temperature free energy of $\mathcal{N} = 4$ super-Yang–Mills theory interpolates smoothly with the ‘t Hooft coupling parameter, barring a possible phase transition between the weak coupling and the strong coupling regimes.

6 Discussion

In this paper, we have explored some aspects of the proposed relation between $d = 4$, $\mathcal{N} = 4$ supersymmetric gauge theory and maximal supergravity on $\mathrm{AdS}_5 \times S_5$ using the Type IIB (p, q) strings as probes. From the point of view of D3-brane and gauge theory, semi-infinite strings attached to it are a natural realization of quarks and anti-quarks. Whether a given configuration involving quarks and anti-quarks is a BPS configuration depends on the relative orientation among the strings (parameterized by angular coordinates on S_5). The physics we have explored, however, did not rely much on it, since the quarks and anti-quarks have an infinite inertia mass and are nominally stable.

The results we have obtained may be summarized as follows. For a single quark Q (or anti-quark $\overline{Q}$) BPS configuration, near-extremal excitation corresponds to fluctuation of the fundamental string. We have found that the governing equations and boundary conditions do match precisely the large-N gauge theory and the anti-de Sitter supergravity sides. In due course, we have clarified the emergence of Polchinski’s D-brane boundary condition (Dirichlet for perpendicular and Neumann for parallel directions) as the limit $\lambda_\mathrm{IIB} \to 0$ is taken. For a non-BPS $Q\overline{Q}$ pair configuration, we first have studied the inter-quark potential and again have found agreement between the gauge theory and the anti-de Sitter supergravity results. Measured in units of the Higgs expectation value, the potential exhibits a linear potential that allows the interpretation of confinement to be made. Because the theory has no mass gap generated by dimensional transmutation, the fact that the string tension is measured in units of the Higgs expectation value may not be so surprising. We have also explored the θ-dependence of the static quark potential by turning on a constant Ramond–Ramond zero-form potential. The $SL(2, \mathbf{Z})$ S-duality of the underlying Type IIB string theory implies immediately that the static quark potential exhibits a cusp behavior at $\theta = \pi$. The potential strength is the weakest at this point and hints at a possible realization of the deconfinement transition at $\theta = \pi$. We also discussed qualitatively two related issues. Via conformal invariance we have pointed out that a static quark configuration can be transformed into an accelerating (or decelerating) configuration. Viewing this as a physical realization of deforming the Wilson loop, we have conjectured that it is this conformal invariance that allows one to prove the large-N Wilson loop equation for a conformally invariant super-Yang–Mills theory. We also argued that the analogs of static baryons $(Q \cdots Q)$ in QCD are represented by multi-prong string junctions.

We think that the results in the present paper may be of some help eventually in understanding dynamical issues in the large-N limit of superconformal gauge theories. For one thing, it would be very interesting to understand the dynamical light or massless quarks and physical excitation spectra. While we have indicated that a qualitative picture of the excitation spectrum as conjectured by Maldacena would follow from a near-extremal excitation of the fundamental strings themselves, a definitive answer awaits for a full-fledged study.

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